

Applications of Super-Energy Tensors

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In this contribution I intend to give a summary of the new relevant results obtained by using the general superenergy tensors. After a quick review of the definition and properties of these tensors, several of their mathematical and physical applications are presented. In particular, their interest and usefulness is mentioned or explicitly analyzed in 1) the study of causal propagation of general fields; 2) the existence of an infinite number of conserved quantities in Ricci-flat spacetimes; 3) the different gravitational theories, such as Einstein's General Relativity or, say, $n = 11$ supergravity; 4) the appearance of some scalars possibly related to entropy or quality factors; 5) the possibility of superenergy exchange between different physical fields and the appearance of mixed conserved currents.

1 Introduction

In last year ERE-98 meeting, I presented what seemed to be the proper universal generalization of the traditional Bel and Bel-Robinson super-energy (s-e) tensors.¹ Since then, a new interest on this subject has developed, as can be checked in this volume with the contributions by Balfagón and Jaén, Bergqvist, Pozo and Parra, and Teyssandier. In this contribution, I would like to present some of the new relevant applications and results derived lately by using super-energy tensors, as well as some of the future developments which are under current consideration. The only pre-requisite to define the s-e tensors is an n -dimensional manifold endowed with a metric of Lorentzian signature. Thus, all the general results are applicable to most known theories (including or not the gravitational field), and some simple examples will be treated here, such as Special and General Relativity, and $n = 11$ supergravity, from where one can also go to, say, type IIA string theory. To that end, I will briefly summarize the definition and properties of the general super-energy tensors in the next subsection. For a much more detailed and comprehensive view, see².

1.1 Definition and properties of the s-e tensors

Let V_n be any differentiable n -dimensional manifold endowed with a metric g of Lorentzian signature $(-, +, \dots, +)$. Indices in V_n run from 0 to $n - 1$ and are denoted by Greek small letters. A useful operation in s-e studies is the standard Hodge dual, denoted by $*$, whose definition and properties can be found at full extent in².

The basic idea is to consider any m -covariant tensor $t_{\mu_1 \dots \mu_m}$ as a so-called r -fold (n_1, \dots, n_r) -form, denoted schematically by $t_{[n_1], \dots, [n_r]}$, where each of the $[n_\Upsilon]$ indicates a block with n_Υ antisymmetrical indices ($\Upsilon = 1, \dots, r$). Several examples of r -fold forms are: $F_{\mu\nu} = F_{[\mu\nu]}$ is a simple 2-form, while $\nabla_\rho F_{\mu\nu}$ is a double (1,2)-form; the Riemann tensor is a double *symmetrical* (2,2)-form (the pairs can be interchanged) and the Ricci tensor is a double *symmetrical* (1,1)-form; a tensor such as $t_{\mu\nu\rho} = t_{(\mu\nu\rho)}$ is a triple *symmetrical* (1,1,1)-form, etcetera.

Then, one can define all possible duals by using the $*$ acting on each of the $[n_\Upsilon]$ blocks, obtaining a total of 2^r different tensors (including $t_{[n_1],\dots,[n_r]}$), each of which is an r -fold form (except when $n_\Upsilon = n$ for some Υ , but these special cases will be treated as self-evident). All these tensors allow to construct a canonical “electric-magnetic” decomposition of $t_{\mu_1\dots\mu_m}$ relative to any observer by means of contraction of each of these duals on all their r blocks with the timelike unit vector \vec{u} ($u_\mu u^\mu = -1$) describing the observer, see^{1,2}: whenever \vec{u} is contracted with a ‘starred’ block $[n - n_\Upsilon]^*$, we obtain a ‘magnetic part’ in that block, and an ‘electric part’ otherwise. Thus, the electric-magnetic parts are (in general) r -fold forms which can be denoted by

$$\begin{aligned} & (\vec{u} \underbrace{EE\dots E}_r)_{[n_1-1],[n_2-1],\dots,[n_r-1]}, (\vec{u} \underbrace{HE\dots E}_{r-1})_{[n-n_1-1],[n_2-1],\dots,[n_r-1]}, \dots \\ & \dots, (\vec{u} \underbrace{E\dots EH}_{r-1})_{[n_1],[n_2-1],\dots,[n-n_r-1]}, (\vec{u} \underbrace{HH\dots E}_{r-2})_{[n-n_1-1],[n-n_2-1],\dots,[n_r-1]}, \dots, \\ & (\vec{u} \underbrace{E\dots EHH}_{r-2})_{[n_1-1],\dots,[n-n_{r-1}-1],[n-n_r-1]}, \dots, (\vec{u} \underbrace{HH\dots H}_r)_{[n-n_1-1],[n-n_2-1],\dots,[n-n_r-1]}, \end{aligned}$$

where, for instance,

$$\begin{aligned} & (\vec{u} \underbrace{EE\dots E}_r)_{\mu_2\dots\mu_{n_1},\nu_2\dots\nu_{n_2},\dots,\rho_2\dots\rho_{n_r}} \equiv \tilde{t}_{\mu_1\mu_2\dots\mu_{n_1},\nu_1\nu_2\dots\nu_{n_2},\dots,\rho_1\rho_2\dots\rho_{n_r}} u^{\mu_1} u^{\nu_1} \dots u^{\rho_1}, \\ & (\vec{u} \underbrace{HE\dots E}_{r-1})_{\mu_{n_1+2}\dots\mu_n,\nu_2\dots\nu_{n_2},\dots,\rho_2\dots\rho_{n_r}} \equiv \tilde{t}_{\mu_{n_1+1}\mu_{n_1+2}\dots\mu_n,\nu_1\nu_2\dots\nu_{n_2},\dots,\rho_1\rho_2\dots\rho_{n_r}} u^{\mu_{n_1+1}} u^{\nu_1} \dots u^{\rho_1} \end{aligned}$$

and so on. Here, $\tilde{t}_{[n_1],\dots,[n_r]}$ denotes the tensor obtained from $t_{[n_1],\dots,[n_r]}$ by permutting the indices such that the first n_1 indices are those precisely in the block $[n_1]$, the next n_2 indices are those in the block $[n_2]$, and so on. There are 2^r E-H parts, they are *spatial* relative to \vec{u} in the sense that they are orthogonal to \vec{u} in any index, and all of them determine $t_{\mu_1\dots\mu_m}$ completely. Besides, $t_{\mu_1\dots\mu_m}$ vanishes iff all its E-H parts do.

Now, the definition of basic s-e tensor for $t_{\mu_1\dots\mu_m}$ is:^{1,2}

$$\begin{aligned} T_{\lambda_1\mu_1\dots\lambda_r\mu_r} \{t\} \equiv & \frac{1}{2} \left\{ \left(t_{[n_1],\dots,[n_r]} \times t_{[n_1],\dots,[n_r]} \right)_{\lambda_1\mu_1\dots\lambda_r\mu_r} + \right. \\ & + \left(t_{[n-n_1],\dots,[n_r]}^* \times t_{[n-n_1],\dots,[n_r]}^* \right)_{\lambda_1\mu_1\dots\lambda_r\mu_r} + \dots + \\ & + \dots + \left(t_{[n_1],\dots,[n-n_r]}^* \times t_{[n_1],\dots,[n-n_r]}^* \right)_{\lambda_1\mu_1\dots\lambda_r\mu_r} + \dots \\ & + \dots + \left(t_{[n-n_1],[n-n_2],\dots,[n_r]}^* \times t_{[n-n_1],[n-n_2],\dots,[n_r]}^* \right)_{\lambda_1\mu_1\dots\lambda_r\mu_r} + \dots + \\ & \left. + \dots \dots + \left(t_{[n-n_1],\dots,[n-n_r]}^* \times t_{[n-n_1],\dots,[n-n_r]}^* \right)_{\lambda_1\mu_1\dots\lambda_r\mu_r} \right\} \quad (1) \end{aligned}$$

where the \times -product is defined for any r -fold (n_1, \dots, n_r) -form by contracting all indices but one of each block in the product of \tilde{t} with itself, that is to say

$$(t \times t)_{\lambda_1\mu_1\dots\lambda_r\mu_r} \equiv \left(\prod_{\Upsilon=1}^r \frac{1}{(n_\Upsilon - 1)!} \right) \tilde{t}_{\lambda_1\rho_2\dots\rho_{n_1},\dots,\lambda_r\sigma_2\dots\sigma_{n_r}} \tilde{t}_{\mu_1}^{\rho_2\dots\rho_{n_1},\dots,\mu_r^{\sigma_2\dots\sigma_{n_r}}}.$$

The s-e tensor (1) of r -fold forms is therefore a $2r$ -covariant tensor. Notice that any dual of $t_{[n_1],\dots,[n_r]}$ gives rise to the same basic s-e tensor (1). Therefore, one only needs to consider blocks with at most $n/2$ indices if n is even, or $(n-1)/2$ if n is odd.

The main properties of (1) are the following (see² for explicit proofs):

1. Symmetries:

$$T_{\lambda_1 \mu_1 \dots \lambda_r \mu_r} \{t\} = T_{(\lambda_1 \mu_1) \dots (\lambda_r \mu_r)} \{t\}$$

2. If the tensor $t_{[n_1], \dots, [n_r]}$ is symmetric in the interchange of the block $[n_\Upsilon]$ with the block $[n_{\Upsilon'}]$ ($n_\Upsilon = n_{\Upsilon'}$), then the s-e tensor (1) is symmetric in the interchange of the corresponding $(\lambda_\Upsilon \mu_\Upsilon)$ - and $(\lambda_{\Upsilon'} \mu_{\Upsilon'})$ -pairs.
3. If n is even, then the s-e tensor (1) is traceless in any $(\lambda_\Upsilon \mu_\Upsilon)$ -pair with $n_\Upsilon = n/2$.
4. The *super-energy density* of the tensor t relative to the timelike vector \vec{u} is denoted by $W_t(\vec{u})$ and defined by

$$W_t(\vec{u}) \equiv T_{\lambda_1 \mu_1 \dots \lambda_r \mu_r} \{t\} u^{\lambda_1} u^{\mu_1} \dots u^{\lambda_r} u^{\mu_r}.$$

Thus, given any unit timelike \vec{u} , the s-e density is half the sum of the positive squares of all the E-H parts of t relative to \vec{u} , that is

$$W_t(\vec{u}) = \frac{1}{2} \left(\begin{aligned} &(\underbrace{{}^t_{\vec{u}} EE \dots E}_r)^2 + (\underbrace{{}^t_{\vec{u}} HE \dots E}_{r-1})^2 + \dots + (\underbrace{{}^t_{\vec{u}} E \dots EH}_{r-1})^2 + \\ &+ (\underbrace{{}^t_{\vec{u}} HHE \dots E}_{r-2})^2 + \dots + (\underbrace{{}^t_{\vec{u}} E \dots EHH}_{r-2})^2 + \dots + (\underbrace{{}^t_{\vec{u}} HHE \dots H}_r)^2 \end{aligned} \right)$$

Then, we have

$$\begin{aligned} &\forall \text{ timelike } \vec{u}, \quad W_t(\vec{u}) \geq 0, \\ &\{\exists \vec{u} \text{ such that } W_t(\vec{u}) = 0\} \iff T_{\lambda_1 \mu_1 \dots \lambda_r \mu_r} \{t\} = 0 \iff t_{\mu_1 \dots \mu_m} = 0. \end{aligned}$$

5. In fact, the s-e density is half the sum of the squares of all the components of t in any orthonormal basis $\{\vec{e}_\mu\}$ with $\vec{e}_0 = \vec{u}$:

$$W_t(\vec{e}_0) = T_{0 \dots 0} \{t\} = \frac{1}{2} \sum_{\mu_1, \dots, \mu_m=0}^{n-1} |t_{\mu_1 \dots \mu_m}|^2.$$

6. Furthermore²

$$W_t(\vec{u}) \propto \tilde{t}_{\mu_1 \dots \mu_{n_1}, \dots, \rho_1 \dots \rho_{n_r}} \tilde{t}_{\nu_1 \dots \nu_{n_1}, \dots, \sigma_1 \dots \sigma_{n_r}} h^{\mu_1 \nu_1} \dots h^{\mu_{n_1} \nu_{n_1}} \dots h^{\rho_1 \sigma_1} \dots h^{\rho_{n_r} \sigma_{n_r}}$$

where $h_{\mu\nu}(\vec{u}) \equiv g_{\mu\nu} + 2u_\mu u_\nu$.

7. The *super-energy flux vectors* of the tensor t relative to the timelike vector \vec{u} are denoted by ${}^\Upsilon \vec{P}_t(\vec{u})$ and defined by

$${}^\Upsilon P_t^\nu(\vec{u}) \equiv -T_{\lambda_1 \mu_1 \dots \lambda_{\Upsilon-1} \mu_{\Upsilon-1}}{}^\nu{}_{\mu_\Upsilon \dots \lambda_r \mu_r} \{t\} u^{\lambda_1} u^{\mu_1} \dots u^{\lambda_{\Upsilon-1}} u^{\mu_{\Upsilon-1}} u^{\mu_\Upsilon} \dots u^{\lambda_r} u^{\mu_r}.$$

The s-e flux vectors can be decomposed with respect to a unit \vec{u} into their timelike component and the corresponding spatial part as

$${}^\Upsilon P_t^\nu(\vec{u}) = W_t(\vec{u}) u^\nu + \left(\delta_\rho^\nu + u^\nu u_\rho \right) {}^\Upsilon P_t^\rho(\vec{u}).$$

8. $\Upsilon \vec{P}_t(\vec{u})$ are causal vectors with the same time orientation than \vec{u} .
9. Some of the above relations for the s-e density and s-e flux vectors are simple particular cases of a completely general and much more important property of the basic s-e tensors (1), generalizing the dominant energy condition for energy-momentum tensors³ and called the *dominant super-energy property* (DSEP), which reads^{1,2,4,5,6}

$$T_{\mu_1 \dots \mu_{2r}} \{t\} k_1^{\mu_1} \dots k_{2r}^{\mu_{2r}} \geq 0 \quad (2)$$

for any future-pointing causal vectors $\vec{k}_1, \dots, \vec{k}_{2r}$. In fact, the above inequality (2) is *strict* if all the vectors $\vec{k}_1, \dots, \vec{k}_{2r}$ are timelike. The DSEP (2) is equivalent to saying that, in any orthonormal basis $\{\vec{e}_\nu\}$, the ‘super-energy’ relative to \vec{e}_0 ‘dominates’ over all other components of $T_{\mu_1 \dots \mu_{2r}}$, that is

$$T_{0\dots 0} \geq |T_{\mu_1 \dots \mu_{2r}}| \quad \forall \mu_1, \dots, \mu_{2r} = 0, \dots, n-1.$$

It is important to remark that any other linear combination with different ‘weights’ of the \times -products in definition (1) will result in the lose of the DSEP.² This leads to the uniqueness of the completely symmetric part of (1), see^{1,2}.

1.2 Explicit expressions of s-e tensors in simple cases

For completeness, let us present the s-e tensors of type (1) explicitly for the first numbers r . Starting with a scalar f ($r = 0$) we simply have

$$T\{f\} = \frac{1}{2} f^2.$$

For any given simple p -form $\Sigma_{\mu_1 \dots \mu_p} = \Sigma_{[\mu_1 \dots \mu_p]}$ (case $r = 1$), the definition (1) produces after expanding the duals²

$$T_{\lambda\mu} \left\{ \Sigma_{[p]} \right\} = \frac{1}{(p-1)!} \left(\Sigma_{\lambda\rho_2 \dots \rho_p} \Sigma_{\mu}^{\rho_2 \dots \rho_p} - \frac{1}{2p} g_{\lambda\mu} \Sigma_{\rho_1 \rho_2 \dots \rho_p} \Sigma^{\rho_1 \rho_2 \dots \rho_p} \right). \quad (3)$$

Consider now any double (p, q) -form $K_{[p],[q]}$ (case $r = 2$) and take its corresponding $\tilde{K}_{[p],[q]}$ with ordered indices: $\tilde{K}_{\mu_1 \dots \mu_p, \nu_1 \dots \nu_q} = \tilde{K}_{[\mu_1 \dots \mu_p], [\nu_1 \dots \nu_q]}$. Its s-e tensor (1) reads²

$$\begin{aligned} T_{\alpha\beta\lambda\mu} \left\{ K_{[p],[q]} \right\} = & \frac{1}{(p-1)!(q-1)!} \left(\tilde{K}_{\alpha\rho_2 \dots \rho_p, \lambda\sigma_2 \dots \sigma_q} \tilde{K}_{\beta}^{\rho_2 \dots \rho_p, \mu}{}^{\sigma_2 \dots \sigma_q} + \right. \\ & + \tilde{K}_{\alpha\rho_2 \dots \rho_p, \mu\sigma_2 \dots \sigma_q} \tilde{K}_{\beta}^{\rho_2 \dots \rho_p, \lambda}{}^{\sigma_2 \dots \sigma_q} - \frac{1}{p} g_{\alpha\beta} \tilde{K}_{\rho_1 \rho_2 \dots \rho_p, \lambda\sigma_2 \dots \sigma_q} \tilde{K}^{\rho_1 \rho_2 \dots \rho_p, \mu}{}^{\sigma_2 \dots \sigma_q} - \\ & \left. - \frac{1}{q} g_{\lambda\mu} \tilde{K}_{\alpha\rho_2 \dots \rho_p, \sigma_1 \sigma_2 \dots \sigma_q} \tilde{K}_{\beta}^{\rho_2 \dots \rho_p, \sigma_1 \sigma_2 \dots \sigma_q} + \frac{1}{2pq} g_{\alpha\beta} g_{\lambda\mu} \tilde{K}_{\rho_1 \rho_2 \dots \rho_p, \sigma_1 \sigma_2 \dots \sigma_q} \tilde{K}^{\rho_1 \rho_2 \dots \rho_p, \sigma_1 \sigma_2 \dots \sigma_q} \right) \end{aligned} \quad (4)$$

One can find similar explicit formulas for the basic s-e tensor $T_{\alpha\beta\lambda\mu\tau\nu} \left\{ A_{[p],[q],[s]} \right\}$ of general triple (p, q, s) -forms,² and so on. It is noteworthy that expressions (3) and (4) do not depend on the dimension n explicitly. This is in fact a general property, so that a formula for the s-e tensor (1) can be given, in general, without any explicit dependence on n . However, for $n = 4$, the above s-e tensors adopt a very simple and illuminating explicit expression by using spinors, see⁵. Another simple expression in general n can be produced by using Clifford algebra techniques, see Pozo and Parra’s contribution.⁷

2 Applications of super-energy tensors

One of the main applications of s-e tensors is to the study of the causal propagation of general fields. In this study, the use of the DSEP is very helpful, and thus very simple conditions on the divergence of the s-e tensor of a field can be found for the field to propagate causally. We refer the reader to ^{6,4} and to Bergqvist's contribution to this volume. Other important applications concern the existence of Bel-Robinson-type tensors for physical fields other than gravity, to the definition of scalars of mathematical interest, and to the existence of new conservation laws. These are considered briefly in what follows.

2.1 The massive scalar field: (super)^k-energy tensors

Consider now a scalar field ϕ with mass m (the massless case is included by setting $m = 0$) satisfying the Klein-Gordon equation

$$\nabla_\rho \nabla^\rho \phi = m^2 \phi. \quad (5)$$

Its energy-momentum tensor reads

$$T_{\lambda\mu} = \nabla_\lambda \phi \nabla_\mu \phi - \frac{1}{2} g_{\lambda\mu} \nabla_\rho \phi \nabla^\rho \phi - \frac{1}{2} g_{\lambda\mu} m^2 \phi^2 \quad (6)$$

which is symmetric and identically divergence-free when (5) holds. Actually, it is remarkable that (6) can be written as

$$T_{\lambda\mu} = T_{\lambda\mu} \left\{ \nabla_{[1]} \phi \right\} + T \{ m \phi \} (-g_{\lambda\mu}).$$

This procedure is systematic ² and one can produce tensors with a higher number of indices by constructing the s-e tensors of type (1) for the higher-order derivatives of ϕ . For instance, one can consider the next step by using $\nabla_{[1]} \nabla_{[1]} \phi$ as starting object and adding the corresponding mass terms: ²

$$\begin{aligned} \mathcal{S}_{\alpha\beta\lambda\mu} \equiv & T_{\alpha\beta\lambda\mu} \left\{ \nabla_{[1]} \nabla_{[1]} \phi \right\} + T_{\alpha\beta} \left\{ \nabla_{[1]} (m\phi) \right\} (-g_{\lambda\mu}) + T_{\lambda\mu} \left\{ m \left(\nabla_{[1]} \phi \right) \right\} (-g_{\alpha\beta}) + \\ & + T \{ m m \phi \} (-g_{\alpha\beta}) (-g_{\lambda\mu}) = 2 \nabla_\alpha \nabla_{(\lambda} \phi \nabla_{\mu)} \nabla_\beta \phi - g_{\alpha\beta} \left(\nabla_\lambda \nabla^\rho \phi \nabla_\mu \nabla_\rho \phi + m^2 \nabla_\lambda \phi \nabla_\mu \phi \right) \\ & - g_{\lambda\mu} \left(\nabla_\alpha \nabla^\rho \phi \nabla_\beta \nabla_\rho \phi + m^2 \nabla_\alpha \phi \nabla_\beta \phi \right) + \frac{1}{2} g_{\alpha\beta} g_{\lambda\mu} \left(\nabla_\sigma \nabla_\rho \phi \nabla^\sigma \nabla^\rho \phi + 2 m^2 \nabla_\rho \phi \nabla^\rho \phi + m^4 \phi^2 \right) \end{aligned} \quad (7)$$

which satisfies

$$\mathcal{S}_{\alpha\beta\lambda\mu} = \mathcal{S}_{(\alpha\beta)(\lambda\mu)} = \mathcal{S}_{\lambda\mu\alpha\beta}. \quad (8)$$

The tensor (7) has been previously considered by Bel (unpublished) and Teyssandier ⁸ for the case of Special Relativity ($n = 4$). The resemblance of this tensor with the Bel tensor (see ^{9,10,11} and the next subsection) has led to consider it as a *super-energy tensor* of the scalar field, describing similar properties as the Bel-Robinson tensor does for the gravitational field. In fact, this correspondence can be sustained on mathematical and physical grounds ², and moreover it can be carried out further to a cascade of infinite (super)^k-energy tensors, one for each natural number k , by considering the $(k+1)^{th}$ covariant derivative of the scalar field as starting object.² An important result is that the (super)^k-energy (tensor) of ϕ vanishes at a point $x \in V_n$ if and only if ϕ and all its derivatives up to the $(k+1)^{th}$ order are zero at x . In particular, *all* these (super)^k-energy tensors vanish in a domain $D \subseteq V_n$ if ϕ vanishes in D .

Even more interesting results can be drawn from the study of the divergence of (7). A straightforward calculation leads to²

$$\begin{aligned}\nabla_\alpha \mathcal{S}^\alpha_{\beta\lambda\mu} = & 2\nabla_\beta \nabla_{(\lambda} \phi R_{\mu)\rho} \nabla^\rho \phi - g_{\lambda\mu} R^{\sigma\rho} \nabla_\beta \nabla_\rho \phi \nabla_\sigma \phi - \\ & - \nabla_\sigma \phi \left(2\nabla^\rho \nabla_{(\lambda} \phi R_{\mu)\rho\beta}^\sigma + g_{\lambda\mu} R_{\rho\beta\tau}^\sigma \nabla^\rho \nabla^\tau \phi \right)\end{aligned}\quad (9)$$

so that it is immediate to see that \mathcal{S} is divergence-free in *flat spacetimes*, which leads to the existence of conserved quantities for the scalar field in the absence of curvature. For a study of these in Special Relativity see⁸. In fact, the above divergence-free result holds for all (super)^k-energy tensors, leading to an infinite number of conserved quantities in flat spacetimes.^{2,12}

2.2 The gravitational field: generalized Bel and Bel-Robinson tensors

Assume that the gravitational field is described by the Riemann tensor $R_{\alpha\beta\lambda\mu}$ of the spacetime. As this is a double symmetric (2,2)-form, the basic s-e tensor for the gravitational field is simply given by the appropriate restriction of (4)

$$\begin{aligned}B_{\alpha\beta\lambda\mu} \equiv T_{\alpha\beta\lambda\mu} \left\{ R_{[2],[2]} \right\} = & R_{\alpha\rho,\lambda\sigma} R_{\beta}^{\rho,\mu\sigma} + R_{\alpha\rho,\mu\sigma} R_{\beta}^{\rho,\lambda\sigma} - \\ & - \frac{1}{2} g_{\alpha\beta} R_{\rho\tau,\lambda\sigma} R^{\rho\tau,\mu\sigma} - \frac{1}{2} g_{\lambda\mu} R_{\alpha\rho,\sigma\tau} R_{\beta}^{\rho,\sigma\tau} + \frac{1}{8} g_{\alpha\beta} g_{\lambda\mu} R_{\rho\tau,\sigma\nu} R^{\rho\tau,\sigma\nu}\end{aligned}\quad (10)$$

which is a straightforward generalization of the original definition^{9,10} of the so-called Bel tensor (see¹³ for $n = 4$, and also¹⁴). The properties of (10) are

$$B_{\alpha\beta\lambda\mu} = B_{(\alpha\beta)(\lambda\mu)} = B_{\lambda\mu\alpha\beta}. \quad (11)$$

and also

$$\nabla_\alpha B^{\alpha\beta\lambda\mu} = R_{\rho}^{\beta\lambda} J^{\mu\sigma\rho} + R_{\rho}^{\beta\mu} J^{\lambda\sigma\rho} - \frac{1}{2} g^{\lambda\mu} R_{\rho\sigma\gamma}^{\beta} J^{\sigma\gamma\rho} \quad (12)$$

where $J_{\lambda\mu\beta} = -J_{\mu\lambda\beta} \equiv \nabla_\lambda R_{\mu\beta} - \nabla_\mu R_{\lambda\beta}$. Thus, B is *divergence-free* when the ‘current’ of matter $J_{\lambda\mu\beta}$ vanishes.

One can also construct the basic s-e tensor for the Weyl tensor, which generalizes the classical Bel-Robinson tensor^{15,16,11} constructed in General Relativity, giving

$$\begin{aligned}\mathcal{T}_{\alpha\beta\lambda\mu} \equiv T_{\alpha\beta\lambda\mu} \left\{ C_{[2],[2]} \right\} = & C_{\alpha\rho,\lambda\sigma} C_{\beta}^{\rho,\mu\sigma} + C_{\alpha\rho,\mu\sigma} C_{\beta}^{\rho,\lambda\sigma} - \\ & - \frac{1}{2} g_{\alpha\beta} C_{\rho\tau,\lambda\sigma} C^{\rho\tau,\mu\sigma} - \frac{1}{2} g_{\lambda\mu} C_{\alpha\rho,\sigma\tau} C_{\beta}^{\rho,\sigma\tau} + \frac{1}{8} g_{\alpha\beta} g_{\lambda\mu} C_{\rho\tau,\sigma\nu} C^{\rho\tau,\sigma\nu}\end{aligned}\quad (13)$$

from where one deduces the same symmetry properties as in (11). One is used to the properties that the Bel-Robinson tensor is completely symmetric and traceless, but these depend on the dimension n of the spacetime. In fact, it can be proved² that (13) is completely symmetric if and only if $n = 4, 5$, and traceless iff $n = 4$. Nevertheless, the divergence of (13) vanishes in general n -dimensional Einstein spaces.

The difference between the Riemann and Weyl tensors is described by the Ricci tensor, which is usually related to the matter contents of the spacetime (as in General Relativity through Einstein’s equations). Thus, one can define a *pure matter* gravitational s-e tensor by means of²

$$\mathcal{M}_{\alpha\beta\lambda\mu} \equiv T_{\alpha\beta\lambda\mu} \left\{ R_{[2],[2]} - C_{[2],[2]} \right\}, \quad \mathcal{M}_{\alpha\beta\lambda\mu} = \mathcal{M}_{(\alpha\beta)(\lambda\mu)} = \mathcal{M}_{\lambda\mu\alpha\beta}$$

which has the interesting property of vanishing iff the spacetime is Ricci flat. Taking into account the classical decomposition of the Riemann tensor into the Weyl tensor and the Ricci terms, one can easily prove

$$B_{\alpha\beta\lambda\mu} = \mathcal{T}_{\alpha\beta\lambda\mu} + \mathcal{M}_{\alpha\beta\lambda\mu} + \mathcal{Q}_{\alpha\beta\lambda\mu}$$

where $\mathcal{Q}_{\alpha\beta\lambda\mu}$ contains the coupled terms and satisfies (for the explicit expression see ²)

$$\mathcal{Q}_{\alpha\beta\lambda\mu} = \mathcal{Q}_{(\alpha\beta)(\lambda\mu)} = \mathcal{Q}_{\lambda\mu\alpha\beta}, \quad \mathcal{Q}_{\alpha(\beta\lambda\mu)} = 0. \quad (14)$$

The last property in (14) allows to prove that, in fact, the tensor \mathcal{Q} does *not* contribute to the generalized Bel s-e flux vector, so that (using obvious notation)

$$\vec{P}_B(\vec{u}) = \vec{P}_T(\vec{u}) + \vec{P}_M(\vec{u}) \quad W_B(\vec{u}) = W_T(\vec{u}) + W_M(\vec{u}),$$

from where one can also easily get a characterization of Ricci-flat spacetimes

$$\begin{aligned} \{\exists \vec{u} \text{ such that } W_M(\vec{u}) = 0\} &\iff \mathcal{M}_{\alpha\beta\lambda\mu} = 0 \iff \\ &\iff R_{\alpha\beta,\lambda\mu} = C_{\alpha\beta,\lambda\mu} \iff R_{\mu\nu} = 0 \iff B_{\alpha\beta\lambda\mu} = \mathcal{T}_{\alpha\beta\lambda\mu}. \end{aligned}$$

The properties of the different gravitational s-e densities allow to compare the relative strength of the Riemann, Weyl and Ricci tensors in a given spacetime. For instance, the three positive scalars relative to an observer \vec{u}

$$q_1 \equiv \frac{W_M(\vec{u})}{W_B(\vec{u})}, \quad q_2 \equiv \frac{W_T(\vec{u})}{W_M(\vec{u})}, \quad q_3 \equiv \frac{W_T(\vec{u})}{W_B(\vec{u})} = q_1 q_2$$

are not independent in general, and they satisfy²

$$0 \leq q_1 \leq 1, \quad 0 \leq q_2 \leq \infty, \quad 0 \leq q_3 \leq 1.$$

Thus, q_2 can be used to analyze the so-called Penrose's Weyl tensor hypothesis^{17,18} and similar conjectures concerning the entropy of the gravitational field¹⁹. On the other hand, q_1 may be a 'quality factor' for approximate solutions of some field equations following the ideas in²⁰. Finally, the scalar q_3 vanishes if and only if the metric is conformally flat, so that q_3 measures the departure from this condition somehow.

Similarly to the case of the scalar field, one can also define the (super)^k-energy tensors for the gravitational field by using the $(k-1)^{th}$ covariant derivative of the Riemann tensor as starting object. These tensors are related for each k with the corresponding k -level of the scalar and other fields² and may have some relevance at points of V_n where the Riemann tensor vanishes but such that some of its derivatives do not (so that every neighbourhood of the point has non-zero curvature). Moreover, the gravitational (super)^k-energy (tensor) vanishes in a domain $D \subseteq V_n$ iff the $(k-1)^{th}$ covariant derivative of the Riemann tensor is zero in D . In particular, *all* gravitational (super)^k-energy tensors vanish in flat regions of (V_n, g) .

All this provides an intrinsic characterization of the (V_n, g) of constant curvature as those with vanishing gravitational (super)²-energy (tensor), so that de Sitter and anti-de Sitter spacetimes can be defined as those spacetimes having identically vanishing (super)²-energy but non-zero super-energy.

2.3 $n=11$ Supergravity

As a simple example of the potentialities of the s-e construction, let us apply it to the now fashionable 11-dimensional Supergravity (11-SUGRA), considering only the bosonic sector for simplicity. As is known, see e.g. ²¹, this includes only two fields, the gravitational one described by the Riemann tensor as usual, and a gauge field $F_{\alpha\beta\gamma\delta} = F_{[\alpha\beta\gamma\delta]}$ which is a simple 4-form. Therefore, one can define the s-e tensor for 11-SUGRA as the generalized Bel tensor (10) for $n = 11$ combined with the s-e tensor constructed for the double (1,4)-form $\nabla_{[1]}F_{[4]}$, which using (4) reads

$$T_{\alpha\beta\lambda\mu} \left\{ \nabla_{[1]}F_{[4]} \right\} = \frac{1}{3!} \left(\nabla_\alpha F_{\lambda\rho\sigma\tau} \nabla_\beta F_\mu^{\rho\sigma\tau} + \nabla_\alpha F_{\mu\rho\sigma\tau} \nabla_\beta F_\lambda^{\rho\sigma\tau} - \right. \\ \left. - g_{\alpha\beta} \nabla_\nu F_{\lambda\rho\sigma\tau} \nabla^\nu F_\mu^{\rho\sigma\tau} - \frac{1}{4} g_{\lambda\mu} \nabla_\alpha F_{\nu\rho\sigma\tau} \nabla_\beta F^{\nu\rho\sigma\tau} + \frac{1}{8} g_{\alpha\beta} g_{\lambda\mu} \nabla_\zeta F_{\nu\rho\sigma\tau} \nabla^\zeta F^{\nu\rho\sigma\tau} \right).$$

Thus, the total s-e tensor for 11-SUGRA should read as

$$T_{\alpha\beta\lambda\mu} \{11\text{-SUGRA}\} \equiv B_{\alpha\beta\lambda\mu} + \kappa T_{\alpha\beta\lambda\mu} \left\{ \nabla_{[1]}F_{[4]} \right\}$$

where $\kappa > 0$ is an available positive constant, possibly related to the coupling constant k_{11} , with the appropriate physical units.

From this expression one can also go to type IIA String theory by projecting ²¹, or alternatively one can construct the s-e tensor for IIA String by starting there. In fact, one should compare the different s-e tensors obtained by these two methods. The relevance of these s-e tensors for the string and other higher dimensional theories is under current investigation. Compare with ^{22,23,24,25,26}.

3 Conserved currents for the Einstein-Klein-Gordon case

Perhaps the most important physical application of the s-e tensors is the possibility that arises of exchange of super-energy-momentum quantities between different physical fields. Herein, we will only consider the case of a scalar field minimally coupled to gravity, that is, the case when the Einstein-Klein-Gordon equations hold. These equations can be written in general dimension n as

$$R_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{n-2} m^2 \phi^2 g_{\mu\nu} \quad (15)$$

from where one can deduce the Klein-Gordon equation (5).

Remember that the situation is as follows: the Bel tensor is divergence-free in Ricci-flat spacetimes (that is, if there is no matter and the Einstein equations hold), and the s-e tensor (7) is divergence-free in the absence of curvature, which can be interpreted as absence of gravitational field. These divergence-free properties lead to conserved currents (i.e., divergence-free vector fields) whenever there are symmetries in the spacetime, see ² for a lengthy discussion. Thus, the natural question arises of whether or not one can combine the two s-e tensors to produce a conserved current in the mixed case: when there are both a scalar field and the curvature that it generates. And the answer is, in general, affirmative ^{12,2}.

In order to prove it, let us assume that the spacetime has a Killing vector $\vec{\xi}$. Then, it is known ^{27,28} that

$$\xi^\mu \nabla_\mu \phi = 0, \quad (\text{if } m \neq 0), \\ \xi^\beta \nabla^\rho \phi \nabla_\beta \nabla_\rho \phi = 0. \quad (16)$$

If the scalar field is massless, then in fact one has $\xi^\mu \nabla_\mu \phi = \text{const.}$ see^{27,28}. In any case, (16) always holds. Using (15) one deduces

$$J_{\lambda\mu\beta} = \nabla_\beta \nabla_\lambda \phi \nabla_\mu \phi - \nabla_\beta \nabla_\mu \phi \nabla_\lambda \phi + \frac{2}{n-2} m^2 \phi (g_{\beta\mu} \nabla_\lambda \phi - g_{\beta\lambda} \nabla_\mu \phi)$$

so that contracting (12) and (9) with three copies of $\vec{\xi}$ and using the above one gets

$$\begin{aligned} \xi^\beta \xi^\lambda \xi^\mu \nabla_\alpha B^\alpha{}_{\beta\lambda\mu} &= \nabla_\sigma \phi \left(2 \nabla_\rho \nabla_{(\lambda} \phi R^\sigma{}_{\mu}{}^\rho{}_{\beta)} + g_{(\lambda\mu} R^{\sigma\rho}{}_{\beta)}{}^\tau \nabla_\rho \nabla_\tau \phi \right) \xi^\beta \xi^\lambda \xi^\mu, \\ \xi^\beta \xi^\lambda \xi^\mu \nabla_\alpha \mathcal{S}^\alpha{}_{\beta\lambda\mu} &= -\nabla_\sigma \phi \left(2 \nabla_\rho \nabla_{(\lambda} \phi R^\sigma{}_{\mu}{}^\rho{}_{\beta)} + g_{(\lambda\mu} R^{\sigma\rho}{}_{\beta)}{}^\tau \nabla_\rho \nabla_\tau \phi \right) \xi^\beta \xi^\lambda \xi^\mu \end{aligned}$$

and, in general, neither of these quantities is zero. However, as is obvious

$$\xi^\beta \xi^\lambda \xi^\mu \nabla_\alpha (B^\alpha{}_{\beta\lambda\mu} + \mathcal{S}^\alpha{}_{\beta\lambda\mu}) = 0$$

so that, using (8), (11) and $\nabla_{(\mu} \xi_{\nu)} = 0$ one can finally write^{12,2}

$$\boxed{\nabla_\alpha j^\alpha = 0} \quad j^\alpha \equiv (B^{\alpha\beta\lambda\mu} + \mathcal{S}^{\alpha\beta\lambda\mu}) \xi_\beta \xi_\lambda \xi_\mu.$$

Notice that only the completely symmetric parts of B and \mathcal{S} are relevant here. The importance of this result is that provides *conserved s-e quantities* (via a typical integration of j^α and Gauss' theorem) *proving the exchange of s-e properties* between the gravitational and scalar fields, because neither $B^{\alpha\beta\lambda\mu} \xi_\beta \xi_\lambda \xi_\mu$ nor $\mathcal{S}^{\alpha\beta\lambda\mu} \xi_\beta \xi_\lambda \xi_\mu$ are divergence-free separately in general.

Actually, one can generalize the above result by contracting with different Killing vectors (if they exist).² Furthermore, the interchange of s-e quantities between the electromagnetic and the gravitational field can also be proven by analyzing the propagation of discontinuities along null hypersurfaces^{1,2,12}. In fact, this last analysis shows that the different (super)^k-energy levels give also raise to conserved mixed quantities.

The expressions of the above divergence-free currents for some explicit spacetimes and the physical properties of the mixed conserved quantities thus generated are under current study. All in all, it seems that the s-e tensors defined last year in¹ are leading to interesting results and applications in several disconnected directions, and their future looks very promising.

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